ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Topology and its Applications 154 (2007) 2779–2785

Topology and its Applications

www.elsevier.com/locate/topol

G_δ-additive families in absolute Souslin spaces and Borel measurable selectors ☆

Jiří Spurný

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, Praha 8, 186 75, Czech Republic

Received 6 June 2006; accepted 30 May 2007

Abstract

It is shown that any G_δ-additive family in an absolute Souslin space has a σ-discrete refinement. The existence of a Borel measurable selector for a G_δ-measurable multivalued mapping is obtained as a corollary.

MSC: 54H05; 54E35

Keywords: σ-discrete refinement; Borel-additive family; Measurable selection

1. Introduction

When dealing with a nonseparable metric space, one often encounters a problem whether a given family of sets can be decomposed into countably many discrete pieces. If this is the case, standard methods of descriptive set theory of separable spaces can be applied to get results analogous to the separable ones. In particular, the existence of measurable selectors for multivalued mappings on nonseparable metric spaces often depends upon the existence of a σ-discrete decomposition of some kind (see [12] or [7, Theorem 4.1]).

We briefly mention some results on decomposability of families in metric spaces. As a starting point we can consider the famous theorem by A.H. Stone stating that any open cover of a metric space has a σ-discrete locally finite open refinement (see [2, Theorem 4.4.1]).

The question of decomposability of a family of sets that are not necessarily open is naturally more delicate. R.W. Hansell showed in [6, Theorem 2] that any Souslin-additive disjoint cover of an absolute Souslin metric space is σ-discretely decomposable. (We refer the reader to the next section for definitions of notions not explained here.) By a result of J. Kaniewski and R. Pol (see [12, Theorem 1]), every point-finite Souslin-additive cover of an absolute Souslin space is σ-discretely decomposable.

What is the situation for point-countable families? In this case, it is more appropriate to look for σ-discrete refinements (see [7, p. 366]). By [10, Theorem 3.1(b)], under suitable set theoretical assumptions there exists a point-

☆ This research was supported in part by the grants GAČR 201/03/0935, GAČR 201/03/D120, and in part by the Research Project MSM 1132 00007 from the Czech Ministry of Education.

E-mail address: spurny@karlin.mff.cuni.cz.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2007.05.012

countable Souslin-additive family in a Polish space that is not σ-discretely refinable. On the other hand, R. Pol showed in [13, Theorem 1.3] that any point-countable Borel-additive family of sets in an arbitrary metrizable space admits a σ-discrete refinement provided each member of the family is of weight at most $\aleph_1$. W.G. Fleissner showed in [3] (see also [4, Theorem 3N]) that under an additional axiom of set theory every point-countable Souslin-additive family is σ-discretely refinable. It is an open question whether it can be proved in a standard set theory that any point-countable Borel-additive cover of a complete metric space has a σ-discrete refinement. (We refer the reader to the survey papers [4] and [5].) R.W. Hansell answer this question affirmatively if the family is F_σ-additive (see [7, Theorem 3.3]). He also showed in [7, Example] that an F_σ-additive cover of a complete space need not admit a σ-discrete refinement if the assumption of point-countability is omitted.

However, the methods of [7] does not work for sets of higher Borel classes. Using a different approach we get in this paper the following result on G_δ-additive families of sets.

Let $\mathcal{A}$ be a family of sets in an absolute Souslin space X such that $\bigcup \mathcal{A}'$ is G_δ in X for any countable subfamily $\mathcal{A}'$. Then $\mathcal{A}$ is σ-discretely refinable and G_δ-additive.

The surprising feature of this result is that we require neither point-countability nor G_δ-additivity of $\mathcal{A}$.

As in [7, Theorem 4.1], this is the key tool to get a selection theorem for G_δ-measurable multivalued mappings between metric spaces. The precise theorem is formulated and proved in the last part of the paper.

2. Preliminaries

By a *space* we always mean a metrizable topological space. We write (X, ρ) when a compatible metric is specified.

If $\mathcal{A} = \{A_i\colon i \in I\}$ is a family of sets in a space X, $\mathcal{A}$ is said to be *discrete* if every point $x \in X$ has a neighbourhood meeting at most one member of $\mathcal{A}$. If $I = \bigcup_n I_n$ such that each $\{A_i\colon i \in I_n\}$ is a discrete family, $\mathcal{A}$ is *σ-discrete*.

The family $\mathcal{A}$ is called *σ-discretely decomposable* if there exist sets $A_i(n)$, $i \in I$, $n \in \mathbb{N}$, such that $A_i = \bigcup_n A_i(n)$ and $\{A_i(n)\colon i \in I\}$ is a discrete family for each $n \in \mathbb{N}$.

A family $\mathcal{R} = \{R_j\colon j \in J\}$ is called a *refinement of* $\mathcal{A}$ if $\bigcup \mathcal{R} = \bigcup \mathcal{A}$ and for every $j \in J$ there exists $i \in I$ with $R_j \subset A_i$.

The family $\mathcal{A}$ is said to admit a *σ-discrete refinement* if there exists a refinement of $\mathcal{A}$ that is σ-discrete (we also say that $\mathcal{A}$ is *σ-discretely refinable*).

If $\mathcal{B}$ is a family of sets in X, $\mathcal{A}$ is called *$\mathcal{B}$-additive* if $\bigcup\{A_i\colon i \in I'\} \in \mathcal{B}$ for every $I' \subset I$. If $\bigcup\{A_i\colon i \in I'\} \in \mathcal{B}$ for any countable set $I' \subset I$, $\mathcal{A}$ is said to be *countably $\mathcal{B}$-additive*.

The family $\mathcal{A}$ of sets is *point-finite* (*point-countable*) if $\{i \in I\colon x \in A_i\}$ is finite (countable) for every $x \in X$.

We recall that a set A in a space is an F_σ-set if A can be written as a countable union of closed sets, G_δ-sets are countable intersections of open sets.

A space X is *absolute Souslin* if X is homeomorphic to a Souslin subset of a complete metric space. It follows from [8, Theorem 1.1] and [9, Theorem 4.1] that X is an absolute Souslin space if and only if there exists a complete metric space Y and a continuous mapping f of Y onto X such that f preserves σ-discretely decomposable families.

If $\mathcal{A} = \{A_i\colon i \in I\}$ is a family of sets in a space X and $F \subset X$, we denote by $\mathcal{A}\restriction_F$ the family $\{A_i \cap F\colon i \in I\}$.

We denote by $\{0,1\}^{<\mathbb{N}}$ the space of finite sequences of digits 0 and 1. Let $|s|$ be the length of s. We denote by $\emptyset$ the empty sequence and adopt the convention that the length of the empty sequence is 0. For $s \in \{0,1\}^{<\mathbb{N}}$ and $i \in \{0,1\}$, we write $s^\wedge i$ for the sequence $(s_1, \ldots, s_{|s|}, i)$.

For a sequence σ in the Cantor set $\{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, we write $\sigma\restriction n$ for the finite sequence $(\sigma_1, \ldots, \sigma_n)$. We adopt the convention that $\sigma\restriction 0 = \emptyset$.

Definition 1. Let $\mathcal{A}$ be a family of sets in a space X. We say that $\mathcal{A}$ is *nowhere σ-discretely refinable* if $\bigcup \mathcal{A} \neq \emptyset$ and $\mathcal{A}\restriction_U$ is not σ-discretely refinable for any open $U \subset X$ intersecting $\bigcup \mathcal{A}$.

We recall the following standard lemma.

Lemma 2. *Let $\mathcal{A}$ be a family of sets in a space X and let*

$$G = \bigcup\{U\colon U \text{ open},\ \mathcal{A}\restriction_U \text{ is } \sigma\text{-discretely refinable}\}.$$

Then $\mathcal{A}\restriction_G$ has a σ-discrete refinement and $\mathcal{A}\restriction_{X\setminus G}$ is nowhere σ-discretely refinable, provided $X \setminus G$ is nonempty.

Proof. Let $\mathcal{U}$ be the family of all open sets U such that $\mathcal{A}\restriction_U$ has a σ-discrete refinement. By Stone's theorem, the family $\mathcal{U}$ has a σ-discrete refinement $\mathcal{R}$. Obviously, $\mathcal{A}\restriction_R$ has a σ-discrete refinement for each $R \in \mathcal{R}$. It is straightforward to check that $\mathcal{A}\restriction_G$ is σ-discretely refinable as $G = \bigcup \mathcal{U} = \bigcup \mathcal{R}$.

For the proof of the second assertion, assume that there exists open $U \subset X$ intersecting $X \setminus G$ such that $\mathcal{A}\restriction_{U\cap(X\setminus G)}$ is σ-discretely refinable. Due to the first part and the fact that $U = (U \cap G) \cup (U \setminus G)$, $\mathcal{A}\restriction_U$ is σ-discretely refinable. Hence $U \subset G$, a contradiction. □

3. G_δ-additive families

We start the proof of the main result with the following simple lemma that will be used in the inductive construction in Lemma 4.

Lemma 3. *Let $\mathcal{A}$ be a family of sets in a space X that is not σ-discretely refinable. Let $\{F_n\colon n \in \mathbb{N}\}$ be a family of closed sets covering X. Then there exists $k \in \mathbb{N}$ and a closed set $H \subset F_k$ such that $\mathcal{A}\restriction_H$ is nowhere σ-discretely refinable.*

Proof. Let $\mathcal{A}$ and $\{F_n\colon n \in \mathbb{N}\}$ be as in the premise. Since $\bigcup_n F_n = X$, there exists $k \in \mathbb{N}$ such that $\mathcal{A}\restriction_{F_k}$ is not σ-discretely refinable. We apply Lemma 2 to the family $\mathcal{A}\restriction_{F_k}$ to get the required closed set $H \subset F_k$ with the property that $\mathcal{A}\restriction_H$ is nowhere σ-discretely refinable. □

Lemma 4. *Let $\mathcal{A} = \{A_i\colon i \in I\}$ be a countably G_δ-additive family in a complete space X. Then there exists a nonempty open set $U \subset X$ such that $\mathcal{A}\restriction_U$ has a σ-discrete refinement.*

Proof. Suppose that the assertion is false, i.e., there exists a countably G_δ-additive family $\mathcal{A}$ of sets in a complete space X such that $\mathcal{A}\restriction_U$ is not σ-discretely refinable for each nonempty open set $U \subset X$. Then it follows that all the sets A_i have empty interior.

We fix on X a compatible metric ρ satisfying $\operatorname{diam} X < 1$.

We set $U_\emptyset = F_\emptyset = X$ and find an arbitrary index $i_\emptyset \in I$ such that $U_\emptyset \cap A_{i_\emptyset} \neq \emptyset$. Let $x_\emptyset$ be a point in $U_\emptyset \cap A_{i_\emptyset}$.

For each $s \in \{0,1\}^{<\mathbb{N}}$, we will find a nonempty open set $U_s \subset X$, a nonempty closed set $F_s \subset X$, a point $x_s \in X$ and an index $i_s \in I$ such that the following conditions are satisfied for every $s \in \{0,1\}^{<\mathbb{N}}$ including the empty sequence $\emptyset$:

(i) $\emptyset = \overline{U}_{s\wedge 0} \cap \overline{U}_{s\wedge 1}$, $\overline{U}_{s\wedge 0} \cup \overline{U}_{s\wedge 1} \subset U_s$, $\operatorname{diam} U_s < 2^{-|s|}$;
(ii) $x_s \in F_s \cap A_{i_s}$, $x_{s\wedge 0} = x_s$, $i_{s\wedge 0} = i_s$;
(iii) $F_{s\wedge 0} \cup F_{s\wedge 1} \subset F_s \subset \overline{U}_s$;
(iv) $\mathcal{A}\restriction_{F_s}$ is nowhere σ-discretely refinable;
(v) $F_{s\wedge 1} \cap \bigcup\{A_{i_t}\colon |t| \leqslant |s|\} = \emptyset$.

To start the construction, we set $x_0 = x_\emptyset$ and $i_0 = i_\emptyset$. We find nonempty open sets U_0, U_1 satisfying (i) such that $x_0 \in U_0$. Set $F_0 = F_\emptyset \cap \overline{U}_0$. Then $x_0 \in F_0$ and $\mathcal{A}\restriction_{F_0}$ is nowhere σ-discretely refinable.

As $\mathcal{A}\restriction_{U_1}$ is not σ-discretely refinable, $\mathcal{A}\restriction_{U_1\setminus A_{i_\emptyset}}$ is not σ-discretely refinable as well. In particular, $U_1 \setminus A_{i_\emptyset} \neq \emptyset$. We write $U_1 \setminus A_{i_\emptyset}$ as a union of countably many closed sets H_n, $n \in \mathbb{N}$. By Lemma 3, there exists a closed set $H \subset (U_1 \setminus A_{i_\emptyset})$ such that $\mathcal{A}\restriction_H$ is nowhere σ-discretely refinable. We find $i_1 \in I$ such that A_{i_1} intersects H and select a point $x_1 \in H \cap A_{i_1}$. By setting $F_1 = H$ we complete the first step of the construction.

Let $n \in \mathbb{N}$ and assume that the required objects have been defined for each $s \in \{0,1\}^{<\mathbb{N}}$ with $|s| \leqslant n$. We fix a sequence $s \in \{0,1\}^{<\mathbb{N}}$ with $|s| = n$. Let $x_{s\wedge 0} = x_s$ and $i_{s\wedge 0} = i_s$. We find nonempty open sets $U_{s\wedge 0}$, $U_{s\wedge 1}$ satisfying condition (i) such that $x_{s\wedge 0} \in U_{s\wedge 0}$ and $U_{s\wedge 1} \cap F_s \neq \emptyset$. (We remark that $\mathcal{A}\restriction_{F_s}$ is nowhere σ-discretely refinable by (iv), in particular x_s is not an isolated point of F_s. Hence $U_{s\wedge 1}$ can be chosen in a manner described above.) We set $F_{s\wedge 0} = F_s \cap \overline{U}_{s\wedge 0}$. Obviously, $x_{s\wedge 0} \in F_{s\wedge 0}$ and $\mathcal{A}\restriction_{F_{s\wedge 0}}$ is nowhere σ-discretely refinable by the inductive assumption.

Set

$$D = U_{s\wedge 1} \cap F_s.$$

Since $\mathcal{A}\restriction_D$ is not σ-discretely refinable by (iv),

$$D \setminus \bigcup\{A_{i_t}\colon |t| \leqslant n\} \neq \emptyset,$$

and moreover,

$$\mathcal{A}\restriction_{D\setminus\bigcup\{A_{i_t}\colon |t|\leqslant n\}}$$

has no σ-discrete refinement as well. Since

$$D \setminus \bigcup\{A_{i_t}\colon |t| \leqslant n\}$$

is an F_σ-set, Lemma 3 yields the existence of a closed set

$$H \subset D \setminus \bigcup\{A_{i_t}\colon |t| \leqslant n\} \tag{1}$$

such that $\mathcal{A}\restriction_H$ is nowhere σ-discretely refinable. We pick an index $i_{s^\wedge 1}$ such that $A_{i_{s^\wedge 1}} \cap H \neq \emptyset$ and choose an arbitrary point $x_{s^\wedge 1} \in H \cap A_{i_{s^\wedge 1}}$. Let $F_{s^\wedge 1} = H$. Since

$$F_{s^\wedge 1} \cap \bigcup\{A_{i_t}\colon |t| \leqslant n\} = \emptyset$$

by (1), the inductive step is finished.

Since X is complete, conditions in (i) ensure that the set

$$C = \bigcap_{n=0}^{\infty} \bigcup_{|s|=n} U_s$$

is a well defined homeomorphic copy of the Cantor set $\{0,1\}^{\mathbb{N}}$. Obviously,

$$C = \bigcup_{\sigma\in\{0,1\}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} U_{\sigma\restriction n}$$

and each $\sigma \in \{0,1\}^{<\mathbb{N}}$ defines a unique point $x_\sigma = \bigcap_n U_{\sigma\restriction n}$.

Let

$$A = \{x_s\colon s \in \{0,1\}^{<\mathbb{N}}\}.$$

(We remark that $x_s = x_\sigma$ if $s \in \{0,1\}^{<\mathbb{N}}$ and $\sigma \in \{0,1\}^{\mathbb{N}}$ are sequences satisfying $\sigma = (s_1, \ldots, s_{|s|}, 0, 0, \ldots)$.)

We need the following couple of claims.

Claim 4.1. *Let $\sigma \in \{0,1\}^{\mathbb{N}}$ and $s \in \{0,1\}^{<\mathbb{N}}$ satisfy $\sigma \restriction |s| = s$. Then $x_\sigma \in F_s$.*

Proof. Let σ and s be as in the premise. By condition (iii),

$$F_{\sigma\restriction n} \subset \overline{U}_{\sigma\restriction n}$$

for each $n \in \mathbb{N}$. By virtue of (iii) again,

$$\overline{U}_{\sigma\restriction n} \cap F_s \neq \emptyset$$

for each $n \geqslant |s|$. It follows that

$$F_s \cap \bigcap_{n=0}^{\infty} U_{\sigma\restriction n} = \bigcap_{n=|s|}^{\infty} (F_s \cap \overline{U}_{\sigma\restriction n}) \neq \emptyset,$$

since the latter term is a centered family of closed sets with diameters tending to zero and X is complete. Since $\bigcap_n U_{\sigma\restriction n}$ is a singleton, the conclusion $x_\sigma \in F_s$ follows. □

Claim 4.2. *The following assertions hold true*:

(a) $A \subset \bigcup\{A_{i_s}\colon s \in \{0,1\}^{<\mathbb{N}}\}$;
(b) $(C \setminus A) \cap \bigcup\{A_{i_s}\colon s \in \{0,1\}^{<\mathbb{N}}\} = \emptyset$.

Proof. The assertion (a) is obvious as $x_s \in A_{i_s}$ for each $s \in \{0,1\}^{<\mathbb{N}}$.

For the proof of the second assertion, let $\sigma \in \{0,1\}^{\mathbb{N}}$ be a sequence containing digit 1 infinitely often. Let $\{n_k\}_k$ be an increasing sequence of natural numbers such that $\sigma_{n_k} = 1$. For a given sequence $s \in \{0,1\}^{<\mathbb{N}}$ we choose $k \in \mathbb{N}$ such that $n_k - 1 \geqslant |s|$. By virtue of Claim 4.1 and condition (v),

$$x_\sigma \in F_{(\sigma_1,\ldots,\sigma_{n_k-1},1)} \subset X \setminus A_{i_s}.$$

As $s \in \{0,1\}^{<\mathbb{N}}$ is arbitrary,

$$x_\sigma \in C \setminus \bigcup \{A_{i_s}\colon s \in \{0,1\}^{<\mathbb{N}}\}.$$

This concludes the proof of (b). $\square$

Now we are ready to finish the proof of the lemma. By Claim 4.2,

$$A = C \cap \bigcup \{A_{i_s}\colon s \in \{0,1\}^{<\mathbb{N}}\}.$$

Since $\mathcal{A}$ is a countably G_δ-additive family, A is a G_δ-subset of C. But this contradicts the Baire category theorem as A is also a countable dense subset of C. $\square$

Lemma 5. *Let $\mathcal{A}$ be a countably G_δ-additive family in a complete space X. Then $\mathcal{A}$ is σ-discretely refinable.*

Proof. Let G be the union of all open sets $U \subset X$ such that $\mathcal{A}\restriction_U$ has a σ-discrete refinement. By Lemma 4, G is nonempty. By Lemma 2, $\mathcal{A}\restriction_G$ is σ-discretely refinable. We claim that $G = X$.

Indeed, if $X \setminus G \neq \emptyset$, we can apply Lemma 4 to $\mathcal{A}\restriction_{X\setminus G}$ and obtain a nonempty open set U such that $U \setminus G \neq \emptyset$ and $\mathcal{A}\restriction_{U\setminus G}$ has a σ-discrete refinement. Then $\mathcal{A}$ is σ-discretely refinable on U, a contradiction with $U \setminus G \neq \emptyset$. This completes the proof. $\square$

Now we are ready for the proof of the main result.

Theorem 6. *Let $\mathcal{A}$ be a countably G_δ-additive family in an absolute Souslin space. Then $\mathcal{A}$ is σ-discretely refinable and it is G_δ-additive.*

Proof. Let $\mathcal{A}$ be a countably G_δ-additive family of sets in an absolute Souslin space X. As mentioned in Section 2, there exists a complete space Y and a continuous surjective mapping $f: Y \to X$ such that f preserves σ-discretely decomposable families. Then $\{f^{-1}(A)\colon A \in \mathcal{A}\}$, as a countably G_δ-additive family in Y, has a σ-discrete refinement by Lemma 5. Since f preserves σ-discretely decomposable families, $\mathcal{A}$ is σ-discretely refinable as well.

For the proof of the second assertion, let $J \subset I$ be given. We want to prove that $G = \bigcup\{A_i\colon i \in J\}$ is a G_δ-set in X.

Since $\{A_i\colon i \in J\}$ is countably G_δ-additive, by the first part we know that it has a σ-discrete refinement $\mathcal{R}$.

First we prove that G is a Borel set in X. For any $R \in \mathcal{R}$ we find an index $i(R) \in J$ such that $R \subset A_{i(R)}$. By replacing each R with $\overline{R} \cap A_{i(R)}$ we can assume that $\mathcal{R}$ consists of G_δ-sets. Since

$$G = \bigcup\{A_i\colon i \in J\} = \bigcup \mathcal{R}$$

and $\mathcal{R}$ is a σ-discrete union of G_δ-sets, G itself is a countable union of G_δ-sets. In particular, it is a Borel subset of X.

We want to prove that G is even a G_δ-set in X. Suppose the contrary. Since $X \setminus G$ is an absolute Souslin set, by [11, Theorem 2(d)] there exists a nonempty compact set $C \subset X$ such that

$$\overline{C \cap G} = \overline{C \setminus G} = C$$

and $C \cap G$ is countable. We find a countable set $J_c \subset J$ such that

$$C \cap G = C \cap \bigcup\{A_i\colon i \in J_c\}.$$

Since $\mathcal{A}$ is countably G_δ-additive, $C \cap G$ is a G_δ-set. But this contradicts the Baire category theorem, as $C \cap G$ is a dense countable set in C and C has no isolated points. Hence G is a G_δ-set which finishes the proof. $\square$

Remark 7. A particular case of Theorem 6 for disjoint families was proved by D.K. Burke and R. Pol in [1, Lemma 4.1]. The author would like to thank the referee for attracting his attention to this interesting paper.

4. A Borel measurable selection

As was mentioned in the introduction, the existence of a σ-discrete refinement is usually sufficient for the existence of a measurable selection. The following theorem is a very particular case of this general principle.

Theorem 8. *Let $\Phi : X \to Y$ be a multivalued mapping from an absolute Souslin space X to a complete space Y such that F has nonempty closed values and*

(a) $\Phi^{-1}(U) = \{x \in X\colon \Phi(x) \cap U \neq \emptyset\}$ *is a G_δ-subset of X for any open $U \subset Y$, or*
(b) $\Phi^{-1}(F)$ *is a G_δ-subset of X for any $F \subset Y$ closed.*

Then there exists a Borel measurable mapping $f : X \to Y$ such that $f(x) \in \Phi(x)$ for each $x \in X$.

Proof. First we assume condition (a). We fix on Y a compatible metric such that $\operatorname{diam} Y < 1$, set $\Phi_0 = \Phi$ and $\mathcal{A}_0 = \{X\}$. By induction we construct a sequence $\{\Phi_n\}$ of multivalued maps from X to Y with nonempty values and coverings $\{\mathcal{A}_n\}$ of X such that, for each $n \in \mathbb{N}$,

(i) $\Phi_{n+1}(x) \subset \Phi_n(x)$ for each $x \in X$,
(ii) $\operatorname{diam} \Phi_n(x) < \frac{1}{n}$ for each $x \in X$,
(iii) $\mathcal{A}_{n+1}$ is a disjoint σ-discrete $G_{\delta\sigma}$-additive cover of X (we recall that a set A is of type $G_{\delta\sigma}$ if A is a countable union of G_δ-sets),
(iv) if $A \in \mathcal{A}_n$, $\Phi_n^{-1}(U) \cap A$ is a G_δ-subset of A for any open set $U \subset Y$.

Since both Φ_0 and $\mathcal{A}_0$ satisfy the requirements, we proceed to the inductive step. Assuming that Φ_n and $\mathcal{A}_n$ have been constructed for $n \geqslant 0$, let $\mathcal{U}$ be an open cover of Y consisting of sets of diameter smaller than $\frac{1}{n+1}$.

Let A be a set in $\mathcal{A}_n$. It follows from (iv) that

$$\left\{A \cap \Phi_n^{-1}(U)\colon U \in \mathcal{U}\right\} \tag{2}$$

is a G_δ-additive family of sets in the space A. Since a Borel subset of an absolute Souslin space is an absolute Souslin space as well, Theorem 6 provides a σ-discrete refinement $\mathcal{R}_A$ of the family in (2). By [14, Lemma 3.4], we may assume that $\mathcal{R}_A$ consists of G_δ-subsets of A. Then $\mathcal{R}_A$ is a $G_{\delta\sigma}$-additive family in A.

Let $\mathcal{R}_A = \bigcup_j \mathcal{R}_{A,j}$ such that each $\mathcal{R}_{A,j}$ is a discrete family. Let $\widehat{R}_{A,j} = \bigcup \mathcal{R}_{A,j}$, $j \in \mathbb{N}$. Since $\widehat{R}_{A,j}$'s are G_δ-subsets of A, by setting $R_{A,1} = \widehat{R}_{A,1}$,

$$R_{A,j+1} = \widehat{R}_{A,j+1} \setminus \bigcup_{k=1}^{j} R_{A,k}, \quad j \in \mathbb{N},$$

we get mutually disjoint $G_{\delta\sigma}$-subsets of A. Then

$$\{R \cap R_{A,j}\colon R \in \mathcal{R}_A,\ j \in \mathbb{N}\}$$

is a disjoint σ-discrete $G_{\delta\sigma}$-additive cover of A.

We put all these families together to obtain $\mathcal{A}_{n+1}$, i.e.,

$$\mathcal{A}_{n+1} = \{R \cap R_{A,j}\colon A \in \mathcal{A}_n,\ R \in \mathcal{R}_A,\ j \in \mathbb{N}\}.$$

As condition (iii) holds for $\mathcal{A}_n$, $\mathcal{A}_{n+1}$ is a disjoint σ-discrete $G_{\delta\sigma}$-additive covering of X and it refines $\{\Phi_n^{-1}(U)\colon U \in \mathcal{U}\}$.

For each $B \in \mathcal{A}_{n+1}$ we choose a set $U_B \in \mathcal{U}$ such that $B \subset \Phi_n^{-1}(U_B)$ and set

$$\Phi_{n+1}(x) = \Phi_n(x) \cap U_B \quad \text{if } x \in B,\ B \in \mathcal{A}_{n+1}.$$

Then Φ_{n+1} has nonempty values and conditions (i) and (ii) are satisfied.

If $B \in \mathcal{A}_{n+1}$ and $V \subset Y$ is open,

$$B \cap \Phi_{n+1}^{-1}(V) = B \cap \Phi_n^{-1}(U_B \cap V)$$

is a G_δ-set in B by the inductive assumption. This finishes the construction.

As Φ has closed values and Y is complete,

$$f(x) = \bigcap_{n=1}^{\infty} \overline{\Phi_n(x)}, \quad x \in X,$$

is a well defined selection of Φ. If U is a open subset of Y, let $U_n = \{y \in U\colon \operatorname{dist}(y, Y \setminus U) > \frac{1}{n}\}$. Then

$$f^{-1}(U) = \bigcup_{n=1}^{\infty} \Phi_n^{-1}(U_n)$$

is a $G_{\delta\sigma}$-subset of X by (iii) and (iv). Thus f is Borel measurable, concluding the proof in case Φ satisfies condition (a).

If Φ satisfies condition (b), we follow the previous reasoning almost word by word. Only in the inductive step, if $\mathcal{U}$ is an open cover of Y consisting of sets of diameter smaller than $\frac{1}{n+1}$, we find a refinement of the family $\{\Phi_n^{-1}(\overline{U})\colon U \in \mathcal{U}\}$ and we set $\Phi_n(x) = \Phi_n(x) \cap \overline{U}_B$ if $x \in B$, $B \in \mathcal{A}_{n+1}$.

Hence the proof is finished. □

References

[1] D.K. Burke, R. Pol, Products of Micheal spaces and completely metrizable spaces, Proc. Amer. Math. Soc. 129 (2000) 1535–1544.
[2] R. Engelking, General Topology, Verlag, Berlin, 1989.
[3] W.G. Fleissner, An axiom for nonseparable Borel theory, Trans. Amer. Math. Soc. 251 (1979) 309–328.
[4] D. Fremlin, Measure-additive coverings and measurable selectors, Dissertationes Math. 260 (1987), 116 pp.
[5] R.W. Hansell, A measurable selection and representation theorem in nonseparable spaces, in: Measure Theory, Oberwolfach 1983, in: Lecture Notes in Math., vol. 1089, Springer, Berlin, 1984, pp. 86–94.
[6] R.W. Hansell, Borel measurable mappings for nonseparable metric spaces, Trans. Amer. Math. Soc. 161 (1971) 145–169.
[7] R.W. Hansell, F_σ-set covers of analytic spaces and first class selectors, Proc. Amer. Math. Soc. (2) 96 (1986) 365–371.
[8] R.W. Hansell, Nonseparable analytic metric space and quotient spaces, Topology Appl. 85 (1998) 143–152.
[9] R.W. Hansell, On characterizing non-separable analytic and extended Borel sets as types of continuous images, Proc. London. Math. Soc. 28 (1974) 683–699.
[10] R.W. Hansell, Point-countable Souslin-additive families and σ-discrete reductions, in: General Topology and its Relation to Modern Analysis and Algebra. V. (Proceedings of the Fifth Prague Topological Symposium, 1981), in: Sigma Series in Pure Mathematics, vol. 3, Heldermann Verlag, Berlin, 1983, pp. 254–260.
[11] J.E. Jayne, C.A. Rogers, Borel isomorphism at the first level: Corrigenda at addenda, Mathematika 27 (1980) 236–260.
[12] J. Kaniewski, R. Pol, Borel-measurable selectors for compact-valued mappings in the non-separable case, Bull. Acad. Polon. Sci. 23 (1975) 1043–1050.
[13] R. Pol, Note on decompositions of metrizable spaces II, Fund. Math. (2) 100 (1978) 129–143.
[14] J. Spurný, F_σ-additive families and the invariance of Borel classes, Proc. Amer. Math. Soc. (3) 133 (2004) 905–915.